NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2622

A DESCRIPTION AND A COMPARISON OF CERTAIN NONLINEAR CURVE-FITTING TECHNIQUES, WITH APPLICATIONS TO THE ANALYSIS OF TRANSIENT-RESPONSE DATA

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SUMMARY

Several common methods for curve fitting a set of data by least squares are described and evaluated. The methods are evaluated by applying them to an example taken from aerodynamics: the problem of calculating the stability parameters of an airplane from flight data.

There are other points considered: application of the methods to minimization problems other than curve fitting and the question of convergence to a mere stationary point as opposed to convergence to a minimum.

Finally, several devices which lead to more rapid convergence of the methods are discussed.,

INTRODUCTION

In the determination of stability parameters from flight data by the analysis of transient responses the use of a least-squares process has been suggested (references 1 and 2). If the elevator of the test airplane is pulsed, it is shown in reference 2 that the response in pitching velocity is a nonlinear function[1] of the stability parameters; the pitching velocity is a sum of exponentials, where the exponents and the coefficients of the exponentials are combinations of the stability derivatives of the airplane. It is one of the purposes of this report to show how the stability parameters of an airplane may be calculated

[1]The expression "nonlinear function" as used in this report should not be confused with nonlinear functions (i.e., functions satisfying nonlinear differential equations) usually considered in aeronautical problems. The expression is used here in a different sense; when it is said that the pitching velocity is a nonlinear function of the stability parameters, it is merely meant that the pitching velocity cannot be expressed as a simple sum of the stability parameters multiplied by constants.

from flight data by means of least-squares curve-fitting techniques. A simple least-squares solution of this problem is given by Prony's method, by which a set of data may be fitted to a sum of exponentials (references 1, 2, and reference 3, pp. 369-370). There are many objections to this method, however, foremost among these being the fact that examples have been encountered for which Prony's method fails entirely. Further examples have been met for which Prony's method, while giving an answer to the problem, did not yield a good fit for the data. In general, therefore, when fitting a sum of exponentials, Prony's method, when it works at all, may best be considered to give only a first approximation to the desired parameters.

The process of curve fitting a nonlinear function by least squares may be applied to a wide field of engineering problems and not merely to the calculation of stability parameters. For this reason, the presentation of this report is such that the methods described may be applied to any reasonably smooth functions (say, functions which are twice continuously differentiable). Emphasis, however, is placed on sums of exponentials, since the occurrence of such functions is quite common and since they are believed to be representative of the entire curve-fitting problem.

The general problem has many possible solutions, the classical one being the use of a Taylor's series with all terms of order higher than the first omitted in order to iterate from the first approximation (reference 2 and reference 3, p. 214). In recent years, however, powerful new methods have been devised (references 4, 5, 6, 7, and 8). It is the general purpose of this report to collect these methods under one cover and to apply a relative evaluation to them by using them to solve the same representative problem. The criteria which will be used for this evaluation are the relative amounts of labor involved and the rate of convergence of the iterations.

ANALYSIS

Relation of the Problem to Aerodynamics

It is shown in reference 1 that if $q(t)$ is the response in pitching velocity of an airplane to an elevator deflection $\delta(t)$, then $q(t)$ and $\delta(t)$ are related by the differential equation

$$(D^2 + bD + k)q(t) = (C_1D + C_0)\delta(t) \tag{1}$$

where D is the operator d/dt; b, k, C_1 and C_0 are constants, dependent on the stability derivatives of the airplane. The problem which then arises is that of determining the best values ("best" to be defined in the sequel) of these constants, being given graphical representations of $q(t)$ and $\delta(t)$ obtained from flight data. This problem is solved in reference 2 by fitting $q(t)$ to a function of the form

$$q = \left[A_1 + \frac{C_1\lambda_1+C_0}{\lambda_1-\lambda_2}\int_0^t e^{-\lambda_1\tau}\delta(\tau)d\tau\right]e^{\lambda_1 t} + \left[A_2 + \frac{C_1\lambda_2+C_0}{\lambda_2-\lambda_1}\int_0^t e^{-\lambda_2\tau}\delta(\tau)d\tau\right]e^{\lambda_2 t} \tag{2}$$

which is the general solution of equation (1). In equation (2), A_1 and A_2 represent constants depending on the initial conditions $q(0)$, $(Dq)_{t=0}$, and $\delta(0)$, while λ_1 and λ_2 are the roots of the characteristic equation $x^2 + bx + k = 0$.

The method used in reference 2 (herein called the Taylor's series method) is described in reference 2 and in reference 3, page 214. It consists of finding a first approximation to b, k, C_1 and C_0 by some means and then using a Taylor's expansion, with all terms of order higher than the first omitted, to linearize q and iterate to the best values of the desired constants. It is the object of this report to describe and evaluate other methods as well as the Taylor's series method for fitting nonlinear functions such as the one given in equation (2).

If the input $\delta(t)$ is of the pulse type, that is, if there is a T such that $\delta(t) = 0$ for all $t \geq T$, it follows from equation (2) that for $t \geq T$, $q(t)$ is a sum of exponentials with constant coefficients:

$$q(t) = B_1e^{\lambda_1 t} + B_2e^{\lambda_2 t} \tag{3}$$

where B_1, B_2, λ_1 and λ_2 are constants. The methods described in this report will be applied to the nonlinear function $B_1e^{\lambda_1 t} + B_2e^{\lambda_2 t}$ since this function is considered to be representative of the general type of problem found. That is, an evaluation of the different methods for fitting nonlinear functions based on the application of the methods to a function of this form is believed to be generally valid. A second (and perhaps more important) reason for choosing a sum of exponentials to use as an example is the frequency with which problems involving these functions themselves occur.

Statement of the General Problem and Its Specialization to Sums of Exponentials

To formulate the problem precisely, suppose $q(t, x_1, x_2, \ldots, x_m)$ is some nonlinear function of the independent variable t and the (constant) parameters x_i. Let $q_\epsilon(t)$ be a quantity which is measured at a set $\{t_i\}$, $i=0, 1, \ldots, N$ of $N+1$ values of t. It is then desired to find the best values of the parameters x_k, that is, those values which minimize

$$M = \sum_{i=0}^{N} [q(t_i) - q_\epsilon(t_i)]^2 \tag{4}$$

The problem specifically studied in this report is the following: Let

$$q(t) \equiv B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} \tag{3}$$

where B_1, B_2, λ_1, λ_2 are written instead of x_1, x_2, x_3, x_4, and suppose $q_\epsilon(t)$ is a measured quantity which, theory tells us, should satisfy an identity of the form (3). The data $q_\epsilon(t)$ are subject, however, to experimental error. Let $q_\epsilon(t)$ be measured at the $N+1$ points t_0, t_1, ... , t_N. The problem then becomes that of finding values of the constants B_1, B_2, λ_1, and λ_2 such that

$$M \equiv \sum_{i=0}^{N} \left[B_1 e^{\lambda_1 t_i} + B_2 e^{\lambda_2 t_i} - q_\epsilon(t_i)\right]^2 \tag{4a}$$

is a minimum.

In general, data which fit a sum of two exponentials are oscillatory; that is, the plotted data have the appearance of a damped sine wave. If this is the case, λ_1, λ_2, B_1, and B_2 are complex numbers and, if $\lambda_1 = l + l'i$ and $B_1 = \frac{1}{2}(\beta + \beta' i)$, where $i^2 = -1$, then $\lambda_2 = l - l'i$, $B_2 = \frac{1}{2}(\beta - \beta' i)$ and

$$q(t) = e^{lt}(\beta \cos l't - \beta' \sin l't) \tag{3a}$$

It is more convenient to work with $q(t)$ in this form than in the complex form (3). Applying this notation to equation (4a), we obtain, finally,

$$M = \sum_{i=0}^{N} \left[e^{lt_i}(\beta \cos l't_i - \beta' \sin l't_i) - q_\epsilon(t_i)\right]^2 \tag{4b}$$

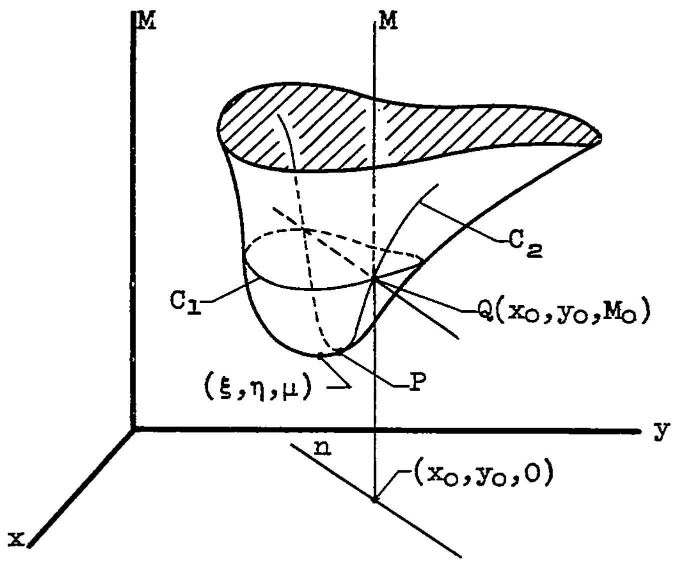
which must be minimized.

The Iteration Methods

It is now assumed that by some means (Prony's method may usually be used for sums of exponentials) a first approximation to the parameters has been found. What follows is a description of the methods which may be used to improve these values.

Steepest descent methods.- In order to be able to apply a geometric interpretation to the usual steepest descent method, it is first assumed that the quantity M to be minimized is a function of only two parameters, x and y (x and y being here written instead of x_1 and x_2

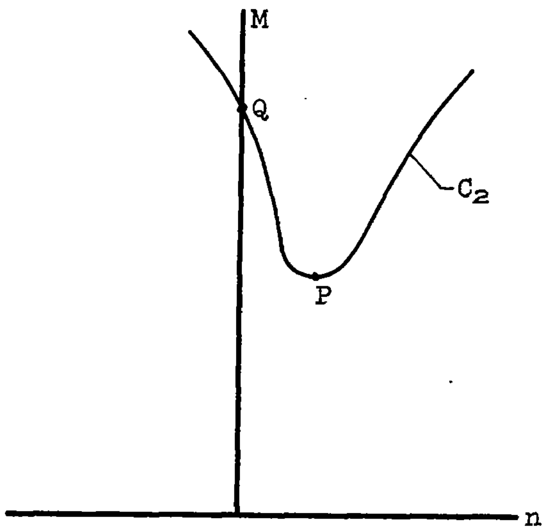
because of the profusion of subscripts which will soon occur). The obvious generalization to more parameters will then be presented. Suppose a first approximation (x_o, y_o) has been found. Let the values of the parameters at which M is a minimum be ξ and η. Further, let M_o, M_{x_o}, M_{y_o}, $M_{x_o x_o}$, $M_{x_o y_o}$, and $M_{y_o y_o}$ be the values of M and its indicated partial derivatives at the first approximation, and let μ be the value of M at (ξ, η). Suppose the surface $M = M(x,y)$ is plotted as in the following sketch:



Sketch (a)

Consider now the level curve C_1, which is the intersection of the surface $M = M(x,y)$ and the plane $M = M_o$. Let Q be the point (x_o,y_o,M_o), and consider also the curve C_2 which is the intersection of $M = M(x,y)$ and the normal plane to C_1 at Q. The direction of steepest descent from the first approximation Q is the direction of the gradient of M, that is, the direction of the curve C_2 at the point Q (reference 9, pp. 76-78). The minimum P of C_2 is taken as a new approximation to (ξ,η,μ) and the process is repeated.

In order to find the coordinates of P, project the normal to the level curve C_1 at (x_o,y_o,M_o) onto the (x,y) plane, and let n be



Sketch (b)

the distance of an arbitrary point on this projection from the point $(x_o,y_o,0)$. A coordinate system has now been introduced in this (M,n) plane. (See sketch (b).)

In this plane, the curve C_2 has some equation $M = M(n)$. By Taylor's theorem,

$$M = M_o + \Delta n\, M_n + \frac{\overline{\Delta n}^2}{2} M_{nn} \tag{5}$$

approximately, where M_n and M_{nn} are the values of the indicated derivatives at the first approximation Q, and Δn denotes a change in the coordinates (x,y) in the direction of the normal to C_1. An approximation to P is to be had by setting $\frac{dM}{d(\Delta n)} = 0$ in equation (5). This procedure leads to

$$\Delta n = -\frac{M_n}{M_{nn}} \tag{6}$$

This value of Δn corresponds to certain increments Δx and Δy. It is these increments which are sought. What follows are manipulations which are needed for their evaluation.

If (n,x) and (n,y) are the angles between the n axis and the x and y axes, respectively, then

$$\cos\,(n,x) = \frac{M_{x_o}}{M_n}\ ;\ \cos\,(n,y) = \frac{M_{y_o}}{M_n}$$

The increments Δx and Δy are given by

$$\left.\begin{aligned} \Delta x &= (\Delta n)\cos\,(n,x) = \frac{M_{x_o}}{M_n}\Delta n \\ \Delta y &= (\Delta n)\cos\,(n,y) = \frac{M_{y_o}}{M_n}\Delta n \end{aligned}\right\} \tag{7}$$

In order to evaluate Δn, expressions for M_n and M_{nn} must be found. An application of vector analysis (reference 9) leads to

$$M_n = \left| \text{grad } M \right| = \sqrt{{M_{x_o}}^2 + {M_{y_o}}^2} \tag{8}$$

It is also clear that

$$M_{nn} = \frac{\partial M_n}{\partial x}\frac{dx}{dn} + \frac{\partial M_n}{\partial y}\frac{dy}{dn} \tag{9}$$

Finally, expressions for dx/dn and dy/dn must be found. The normal to the level curve is the intersection of the two planes $y - y_o = -\left(\frac{dx}{dy}\right)(x - x_o)$ and $M = M_o$, where dx/dy is the reciprocal of the slope of the curve C_1 in the plane $M = M_o$. From this last equation $(M = M_o)$, it follows that $\frac{dx}{dy} = -\frac{M_{y_o}}{M_{x_o}}$. The equations of the n axis of the (M,n) plane in the space (M,x,y) then become

$$\left.\begin{aligned} y - y_o &= \frac{M_{y_o}}{M_{x_o}}(x - x_o) \\ M &= 0 \end{aligned}\right\}$$

Consider a point (x,y) on n. Then $y - y_o = \frac{M_{y_o}}{M_{x_o}}(x-x_o)$. By the definition of n,

$$n = \sqrt{\left(\frac{M_{y_o}}{M_{x_o}}\right)^2(x-x_o)^2 + (x-x_o)^2} = \frac{(x-x_o)\sqrt{{M_{x_o}}^2 + {M_{y_o}}^2}}{M_{x_o}}$$

or

$$x = \frac{nM_{x_o}}{\sqrt{{M_{x_o}}^2 + {M_{y_o}}^2}} + x_o = n\frac{M_{x_o}}{M_n} + x_o$$

Similarly,

$$y = \frac{nM_{y_o}}{\sqrt{{M_{x_o}}^2 + {M_{y_o}}^2}} + y_o = n\frac{M_{y_o}}{M_n} + y_o$$

Therefore

$$\left.\begin{aligned} \frac{dx}{dn} &= \frac{M_{x_o}}{M_n} \\ \frac{dy}{dn} &= \frac{M_{y_o}}{M_n} \end{aligned}\right\} \tag{10}$$

Returning to equation (9), it then follows that

$$M_{nn} = \frac{\left(M_{x_o} M_{x_o x_o} + M_{y_o} M_{x_o y_o}\right)}{M_n} \frac{M_{x_o}}{M_n} + \frac{\left(M_{x_o} M_{x_o y_o} + M_{y_o} M_{y_o y_o}\right)}{M_n} \frac{M_{y_o}}{M_n} \tag{9a}$$

Thus finally, utilizing equations (6), (7), (8), and (9a),

$$\left.\begin{aligned} \Delta x &= -\frac{M_{x_o}\left(M_{x_o}{}^2 + M_{y_o}{}^2\right)}{M_{x_o}\left(M_{x_o} M_{x_o x_o} + M_{y_o} M_{x_o y_o}\right) + M_{y_o}\left(M_{x_o} M_{x_o y_o} + M_{y_o} M_{y_o y_o}\right)} \\ \Delta y &= -\frac{M_{y_o}\left(M_{x_o}{}^2 + M_{y_o}{}^2\right)}{M_{x_o}\left(M_{x_o} M_{x_o x_o} + M_{y_o} M_{x_o y_o}\right) + M_{y_o}\left(M_{x_o} M_{x_o y_o} + M_{y_o} M_{y_o y_o}\right)} \end{aligned}\right\} \tag{7a}$$

which are the desired values of the increments for the case of two parameters.

These equations will now be generalized. Suppose then that M is a function of the m parameters $x_1, x_2, \ldots, x_m$. Let M_o, M_{i_o} denote the values of M and $\partial M/\partial x_i$, respectively, at some first approximation. Equations (8), (9), and (9a) become

$$M_n = \left|\text{grad } M\right| = \left(\sum_{i=1}^{m} M_{i_o}{}^2\right)^{1/2} \tag{11}$$

$$M_{nn} = \sum_{i=1}^{m} \frac{\partial M_n}{\partial x_i} \frac{\partial x_i}{\partial n} = \frac{\sum_{i,j=1}^{m} M_{i_o} M_{j_o} M_{i_o j_o}}{M_n{}^2} \tag{12}$$

and equations (7a) become

$$\Delta x_k = -\frac{M_{k_o}\left(\sum_{i=1}^{m} M_{i_o}{}^2\right)}{\sum_{i,j=1}^{m} M_{i_o} M_{j_o} M_{i_o j_o}} \tag{13}$$

In practice this method would seldom be applied directly to a first approximation. Rather the rougher approximation $\Delta n = -\frac{M_o}{M_n}$, which will herein be called the method of steepest descent along a tangent, after Booth (reference 4), would be used until the value of M begins to increase. In terms of the individual increments, this method leads to

$$\Delta x_k = - \frac{M_o M_{k_o}}{\sum_{i=1}^{m} M_{i_o}^2} \tag{14}$$

Another steepest descent method, one which utilizes quadratic interpolation, was developed by Booth in reference 4. As before, call M_o the value of M at the first approximation. Calculate increments from equations (14). Call M_1 the value of M at the new approximation obtained by adding these increments to the first approximations. Further, let $M_{1/2}$ be the value of M at the parametric values obtained by adding to the first approximations one-half the increments found from equations (14). The increments which Booth finally uses then become

$$\Delta x_k = - \left[\frac{M_1 - 4M_{1/2} + 3M_o}{4(M_1 - 2M_{1/2} + M_o)} \right] \frac{M_o M_{k_o}}{\sum_{i=1}^{m} M_{i_o}^2} \tag{15}$$

The formulas which have been derived will now be explicitly given for the case where q is the sum of two exponentials; that is, the case where

$$M = \sum_{i=o}^{N} \left[e^{l t_i}(\beta \cos l' t_i - \beta' \sin l' t_i) - q_\epsilon(t_i) \right]^2 \tag{4b}$$

Let M_o, M_{l_o}, $M_{l'_o}$, M_{β_o}, and $M_{\beta'_o}$ be the values of M and its derivatives at the first approximations. Then,

$$M_n^2 = M_{l_o}^2 + M_{l'_o}^2 + M_{\beta_o}^2 + M_{\beta'_o}^2 \tag{8a}$$

The method of steepest descent along a tangent gives the incremental values

$$\left. \begin{aligned} \Delta l &= - \frac{M_o M_{l_o}}{M_n^2} \\ \Delta l' &= - \frac{M_o M_{l'_o}}{M_n^2} \\ \Delta \beta &= - \frac{M_o M_{\beta_o}}{M_n^2} \\ \Delta \beta' &= - \frac{M_o M_{\beta'_o}}{M_n^2} \end{aligned} \right\}$$

These increments must be added to the corresponding first approximations to obtain new approximations. This process is then repeated until M increases. The parametric values which gave the smallest value to M are then used as first approximations, and one of the finer methods is applied.

For the ordinary method of steepest descent to a minimum, equations (13) become

$$\left.\begin{aligned} \Delta l &= -\frac{M_{l_o}}{M_{nn}} \\ \Delta l' &= -\frac{M_{l'_o}}{M_{nn}} \\ \Delta \beta &= -\frac{M_{\beta_o}}{M_{nn}} \\ \Delta \beta' &= -\frac{M_{\beta'_o}}{M_{nn}} \end{aligned}\right\} \tag{13a}$$

where M_{nn} is given by

$$M_{nn} = \frac{1}{M_n^2}\left[M_{l_o}^2 M_{l_o l_o} + M_{l'_o}^2 M_{l'_o l'_o} + M_{\beta_o}^2 M_{\beta_o \beta_o} + M_{\beta'_o}^2 M_{\beta'_o \beta'_o} + 2\left(M_{l_o} M_{l'_o} M_{l_o l'_o} + M_{l_o} M_{\beta_o} M_{l_o \beta_o} + M_{l_o} M_{\beta'_o} M_{l_o \beta'_o} + M_{l'_o} M_{\beta_o} M_{l'_o \beta_o} + M_{l'_o} M_{\beta'_o} M_{l'_o \beta'_o} + M_{\beta_o} M_{\beta'_o} M_{\beta_o \beta'_o}\right)\right] \tag{12a}$$

The quantity M_n is displayed in equation (8a). All the derivatives in these equations are to be evaluated at the first approximations.

Finally, Booth's method gives

$$\left.\begin{aligned} \Delta l &= -F\frac{M_o M_{l_o}}{M_n^2} \\ \Delta l' &= -F\frac{M_o M_{l'_o}}{M_n^2} \\ \Delta \beta &= -F\frac{M_o M_{\beta_o}}{M_n^2} \\ \Delta \beta' &= -F\frac{M_o M_{\beta'_o}}{M_n^2} \end{aligned}\right\} \tag{15a}$$

where

$$F = \frac{M_1 - 4M_{1/2} + 3M_o}{4(M_1 - 2M_{1/2} + M_o)} \tag{16}$$

M_1, $M_{1/2}$ and M_o are as defined previously.

For later use, the derivatives needed for these methods are given below. Let

$$\epsilon_i = e^{lt_i}(\beta \cos l't_i - \beta' \sin l't_i) - q_\epsilon(t_i)$$

Then

$$M = \sum_{i=o}^{N} \epsilon_i^{\,2} \tag{4b}$$

and

$$\left.\begin{aligned}
M_l &= 2\sum_{i=o}^{N} \epsilon_i t_i e^{lt_i}(\beta \cos l't_i - \beta' \sin l't_i) \\
M_{l'} &= -2\sum_{i=o}^{N} \epsilon_i t_i e^{lt_i}(\beta \sin l't_i + \beta' \cos l't_i) \\
M_\beta &= 2\sum_{i=o}^{N} \epsilon_i e^{lt_i} \cos l't_i \\
M_{\beta'} &= -2\sum_{i=o}^{N} \epsilon_i e^{lt_i} \sin l't_i
\end{aligned}\right\} \tag{17}$$

The second derivatives are

$$
\left.
\begin{aligned}
M_{ll} &= 2\sum_{i=o}^{N} t_i^2 e^{lt_i}(\beta \cos l't_i - \beta' \sin l't_i) \times \\
&\qquad [\epsilon_i + e^{lt_i}(\beta \cos l't_i - \beta' \sin l't_i)] \\
M_{l'l'} &= 2\sum_{i=o}^{N} [t_i e^{lt_i}(\beta \sin l't_i + \beta' \cos l't_i)]^2 - \\
&\qquad 2\sum_{i=o}^{N} \epsilon_i t_i^2 e^{lt_i}(\beta \cos l't_i - \beta' \sin l't_i) \\
M_{\beta\beta} &= 2\sum_{i=o}^{N} (e^{lt_i} \cos l't_i)^2 \\
M_{\beta'\beta'} &= 2\sum_{i=o}^{N} (e^{lt_i} \sin l't_i)^2
\end{aligned}
\right\} \qquad (18)
$$

while the mixed second derivatives are given by

$$
\left.
\begin{aligned}
M_{ll'} &= -2\sum_{i=0}^{N} t_i^2 e^{lt_i}(\beta \sin l't_i + \beta' \cos l't_i) \times \\
&\qquad [\epsilon_i + e^{lt_i}(\beta \cos l't_i - \beta' \sin l't_i)] \\
M_{l\beta} &= 2\sum_{i=0}^{N} t_i e^{lt_i} \cos l't_i [\epsilon_i + e^{lt_i}(\beta \cos l't_i - \beta' \sin l't_i)] \\
M_{l\beta'} &= -2\sum_{i=0}^{N} t_i e^{lt_i} \sin l't_i [\epsilon_i + e^{lt_i}(\beta \cos l't_i - \beta' \sin l't_i)] \\
M_{l'\beta} &= -2\sum_{i=0}^{N} \epsilon_i t_i e^{lt_i} \sin l't_i - 2\sum_{i=0}^{N} (t_i e^{lt_i} \cos l't_i) \times \\
&\qquad [e^{lt_i}(\beta \sin l't_i + \beta' \cos l't_i)] \\
M_{l'\beta'} &= 2\sum_{i=0}^{N} (t_i e^{lt_i} \sin l't_i)[e^{lt_i}(\beta \sin l't_i + \beta' \cos l't_i)] - \\
&\qquad 2\sum_{i=0}^{N} \epsilon_i t_i e^{lt_i} \cos l't_i \\
M_{\beta\beta'} &= -2\sum_{i=0}^{N} (e^{lt_i} \cos l't_i)(e^{lt_i} \sin l't_i)
\end{aligned}
\right\} \quad (19)
$$

The relaxation method.- An excellent geometric interpretation of the relaxation method in its application to the solution of sets of simultaneous linear equations may be found in reference 5. The formulas required for the solution of the problem posed in this report are derived in reference 4.

Suppose M, the quantity to be minimized, is a function of the m parameters $x_1, x_2, \ldots, x_m$. Assume further that some approximation to those values of the parameters which minimize M has been found. Holding all these variables save one (say, x_k) constant at these approximations, M becomes a function of x_k and, applying Taylor's theorem, we obtain

$$M = M_o + \Delta x_k M_{k_o} + \frac{\overline{\Delta x_k}^2}{2} M_{k_o k_o}$$

using the same notation as before. For M to be a minimum, $\frac{dM}{dx_k} \approx \frac{dM}{d(\Delta x_k)}$ must be zero. That is,

$$\Delta x_k = -\frac{M_{k_o}}{M_{k_o k_o}} \tag{20}$$

As for the question of which of the m parameters to vary, Southwell (reference 6) suggests finding the increment Δx_k in that variable x_k for which $|M_{k_o}|$ is largest. Synge (reference 5) varies x_k, where k is determined as that index which makes $|M_{k_o}{}^2/M_{k_o k_o}|$ largest. Synge's method has the disadvantage of requiring the computation of all the second derivatives $M_{k_o k_o}$ at each iteration. For this reason, Southwell's method will be used in this report.

If the data q_ϵ are to be fitted to a sum of two exponentials, M is given by equation (4b). The derivatives needed for the application of the relaxation method are displayed in equations (17) and (18).

The first step in the application of the relaxation method is the calculation of all the first derivatives of M to ascertain which has the largest numerical value. If M_l (say) is the largest of the first derivatives, then l is changed by an amount $\Delta l = -\frac{M_{l_o}}{M_{l_o l_o}}$. Similar formulas are used for the other parameters.

The Taylor's series method.- This method is described in detail in reference 2. The essential formulas may be easily derived, however, and they will be duplicated here.

Suppose, as before, that q is a function of the m parameters $x_1, x_2, \ldots, x_m$ and the independent variable t:

$$q = q(t, x_1, x_2, \ldots, x_m) \tag{3b}$$

Letting zero subscripts denote the value of the indicated quantity at the first approximation, and applying Taylor's theorem, it may be seen that

$$q \approx q_o + \sum_{k=1}^{m} \left(\frac{\partial q}{\partial x_k}\right)_o \Delta x_k$$

where $\Delta x_k = x_k - (x_k)_o$. In the Taylor's series method, M is not minimized. Rather, the minimization procedure is applied to the approximating function

$$\overline{M} = \sum_{i=o}^{N} \left\{ \left[q_o + \sum_{k=1}^{m} \left(\frac{\partial q}{\partial x_k}\right)_o \Delta x_k \right]_{t=t_i} - q_\epsilon(t_i) \right\}^2 \tag{4c}$$

This minimization leads to the set of (linear) simultaneous equations

$$\frac{\partial \overline{M}}{\partial(\Delta x_k)} = 0, \qquad k = 1, 2, \ldots, m \tag{21}$$

If

$$q = e^{lt}(\beta \cos l't - \beta' \sin l't) \tag{3a}$$

then

$$\left.\begin{aligned}
\frac{\partial q}{\partial l} &= te^{lt}(\beta \cos l't - \beta' \sin l't) \\
\frac{\partial q}{\partial l'} &= -\, te^{lt}(\beta \sin l't + \beta' \cos l't) \\
\frac{\partial q}{\partial \beta} &= e^{lt} \cos l't \\
\frac{\partial q}{\partial \beta'} &= -\, e^{lt} \sin l't
\end{aligned}\right\} \tag{22}$$

These derivatives are reproduced here for use later.

<u>The method of damped least squares and the problem of increasing M.</u>- It may sometimes occur that the least-squares processes which have thus far been described may fail to converge. Upon applying the Taylor's

series method, for example, increments may be found which, although diminishing $\bar{M}$ (equation (4c)), may cause M to increase. A procedure which may be tried when one method fails is to apply one of the other methods, as failure of one method for a particular example does not imply failure of all the others.

Levenberg (reference 7) suggests a different procedure, which he calls the method of damped least squares. In order to keep the increments small, he minimizes

$$M' = w\bar{M} + \sum_{k=1}^{m} w_k(\Delta x_k)^2 \tag{4d}$$

where $\bar{M}$ is given in equation (4c), while w and w_k are weighting factors. A more general procedure than this would be to minimize

$$M'' = wM + \sum_{k=1}^{m} w_k(\Delta x_k)^2 \tag{4e}$$

by any of the methods previously described, where M is given by equation (4).

Completion of the Aerodynamic Problem of the Calculation of Stability Parameters

It was stated in the first section of this report that an aerodynamic problem to which the methods herein described are applicable is that of determining the stability coefficients, b, k, C_1 and C_o, of an airplane. These parameters are constants which occur in the differential equation (1) relating pitching velocity with elevator input. This report has not as yet shown, however, how these constants may be determined; rather it has been shown how the four other numbers l, l', β, and β' may be found. It is the purpose of this section to describe the means by which b, k, C_1 and C_o may be calculated from the knowledge of l, l', β, and β'.

The constants b and k may be immediately computed from the equations

$$\left.\begin{aligned} b &= -2l \\ k &= l^2 + l'^2 \end{aligned}\right\} \tag{23}$$

In reference 2, it was shown how the constants C_1 and C_0 may be found. However, the quite valid objection that the method weights the initial conditions[2] $q(o)$ and $\dot{q}(o)$ too heavily can be raised. Not only is $\dot{q}(o)$ heavily weighted, but it is also true that it is difficult to calculate the derivative $\dot{q}(o)$ at all accurately. The method described below does not involve the calculation of $\dot{q}$ and also, instead of being entirely dependent on the value of q at the initial point, applies a simple least-squares procedure to all the points at which $\delta(t)$ is not yet zero to calculate C_1 and C_0.

Referring to equation (2), the following notation will be used:

$$\left.\begin{aligned} A_1 &= \frac{\alpha}{2} + \frac{\alpha'}{2}i \\ \int_0^{t_k} e^{-\lambda_1\tau}\delta(\tau)d\tau &= \sigma_k - \sigma_k' i \end{aligned}\right\}$$

It is clear, then, that

$$\left.\begin{aligned} A_2 &= \frac{\alpha}{2} - \frac{\alpha'}{2}i \\ \int_0^{t_k} e^{-\lambda_2\tau}\delta(\tau)d\tau &= \sigma_k + \sigma_k' i \end{aligned}\right\}$$

where

$$\left.\begin{aligned} \sigma_k &= \int_0^{t_k} e^{-l\tau}\delta(\tau)\cos l'\tau d\tau \\ \sigma_k' &= \int_0^{t_k} e^{-l\tau}\delta(\tau)\sin l'\tau d\tau \end{aligned}\right\} \tag{24}$$

Also,

$$q(t_k) = e^{lt_k}\left\{\left[\alpha + \frac{(l'\sigma_k - l\sigma_k')C_1 - \sigma_k' C_0}{l'}\right]\cos l't_k - \left[\alpha' - \frac{(l'\sigma_k' + l\sigma_k)C_1 + \sigma_k C_0}{l'}\right]\sin l't_k\right\} \tag{3c}$$

---

[2]A dot is here used to denote differentiation with respect to t.

If the input $\delta(t)$ is a pulse which is zero for all $t \geq t_j$, then, for $t_k \geq t_j$, σ_k and σ_k' are constants. Let

$$\sigma - \sigma' i = \int_0^{t_j} e^{-\lambda_1 \tau} \delta(\tau) d\tau$$

Thus, $\sigma_k = \sigma$ and $\sigma_k' = \sigma'$ for all $t_k \geq t_j$. Further, for all $t_k \geq t_j$,

$$q(t_k) = e^{l t_k}(\beta \cos l' t_k - \beta' \sin l' t_k) \tag{3a}$$

Thus, by comparing equations (3a) and (3c), since α, α', β, and β' are constants,

$$\alpha = \beta - \frac{(l'\sigma - l\sigma')C_1 - \sigma' C_0}{l'}$$

$$\alpha' = \beta' + \frac{(l'\sigma' + l\sigma)C_1 + \sigma C_0}{l'}$$

and, substituting into equation (3c),

$$q(t_k) = e^{l t_k}\left\{\left[\beta - \frac{(l'\sigma - l\sigma')C_1 - \sigma' C_0}{l'} + \frac{(l'\sigma_k - l\sigma_k')C_1 - \sigma_k' C_0}{l'}\right] \cos l' t_k - \right.$$

$$\left. \left[\beta' + \frac{(l'\sigma' + l\sigma)C_1 + \sigma C_0}{l'} - \frac{(l'\sigma_k' + l\sigma_k)C_1 + \sigma_k C_0}{l'}\right] \sin l' t_k\right\}$$

Replacing $q(t_k)$ by $q_\epsilon(t_k)$, it may be found that the following equation is approximately true:

$$\frac{(\sigma' - \sigma_k')e^{l t_k}\cos l' t_k - (\sigma - \sigma_k)e^{l t_k}\sin l' t_k}{l'} C_0 -$$

$$\frac{[l'(\sigma - \sigma_k) - l(\sigma' - \sigma_k')]e^{l t_k}\cos l' t_k + [l'(\sigma' - \sigma_k') + l(\sigma - \sigma_k)]e^{l t_k}\sin l' t_k}{l'} C_1$$

$$= q_\epsilon(t_k) - e^{l t}(\beta \cos l' t_k - \beta' \sin l' t_k) \tag{25}$$

Equations (25) are a set of $(j+1)$ linear equations (for $k = 0, 1, \ldots, j$). Since l, l', β, and β' are known, equations (25) may be solved by the ordinary method of least squares for linear equations (reference 3) for C_1 and C_0.

APPLICATION TO AN EXAMPLE

The First Approximation and the Method of Steepest Descent Along a Tangent

The example to which all methods described in this report will be applied is actual flight data obtained by measuring the pitching velocity of a test airplane in response to an elevator pulse which was zero for all $t \geq 0.4$ ($t = 0$ being taken at the start of the pulse). These data are given on this page.

t	$q_\epsilon(t)$
0.4	0.224
.5	.120
.6	.020
.7	-.057
.8	-.112
.9	-.148
1.0	-.160
1.1	-.150
1.2	-.127
1.3	-.097
1.4	-.062
1.5	-.032
1.6	-.005
1.7	.017
1.8	.030
1.9	.036
2.0	.035
2.1	.032
2.2	.027
2.3	.020
2.4	.015
2.5	.011
2.6	.008
2.7	.005
2.8	.003
2.9	.001
3.0	0
3.1	0
3.2	0

Since $q_\epsilon(t)$ is oscillatory, the exponents λ_i must be complex, and if $\lambda_1 = l + l'i$ and $B_1 = \frac{1}{2}(\beta+\beta'i)$, then $\lambda_2 = l - l'i$ and $B_2 = \frac{1}{2}(\beta - \beta'i)$. Furthermore, $q(t)$ and M are as given in equations (3a) and (4b), respectively.

It is now assumed that a first approximation to the parameters has been found. Prony's method (references 1, 2, and reference 3, pp. 369-370) was applied to the data in the table on this page giving, if zero subscripts denote the values of the indicated parameters at the first approximations,

$$\left.\begin{aligned} l_o &= -1.1660 \\ l_o' &= 3.2700 \\ \beta_o &= .4616 \\ \beta_o' &= -.2450 \end{aligned}\right\} \qquad (26)$$

In many examples which occur in practice, the first approximations which have been found may be rather rough, and a rapid method for improving these values before applying any of the finer methods described previously is desirable. Such a method has been described in the section on steepest descent methods where it was called the method of steepest descent along a tangent. If M_o, M_{l_o}, $M_{l'_o}$, M_{β_o}, and $M_{\beta'_o}$ denote the values of M and its indicated partial derivatives at the first approximations found above, and if

$$M_n^2 = M_{l_o}^2 + M_{l'_o}^2 + M_{\beta_o}^2 + M_{\beta'_o}^2$$

then it was shown in that section that the values of the increments from which new approximations may be obtained are

$$\left.\begin{aligned} \Delta l &= -\frac{M_o M_{l_o}}{M_n^2} \\ \Delta l' &= -\frac{M_o M_{l'_o}}{M_n^2} \\ \Delta \beta &= -\frac{M_o M_{\beta_o}}{M_n^2} \\ \Delta \beta' &= -\frac{M_o M_{\beta'_o}}{M_n^2} \end{aligned}\right\} \tag{14a}$$

This method is applied repeatedly until the value of M increases. The values of the parameters which gave the smallest value of M are then used as new approximations to which are applied one of the finer methods.

The method of steepest descent along a tangent is applied to the data of this report in table I. Referring to table I and letting circled numbers refer to columns, it may be seen that

$$\begin{aligned} M_o &= \Sigma (14)^2 = 0.0028 \\ M_{l_o} &= 2\Sigma (14) \times (15) = 0.0105 \\ M_{l'_o} &= -2\Sigma (14) \times (19) = 0.00911 \\ M_{\beta_o} &= 2\Sigma (8) \times (14) = 0.000246 \\ M_{\beta'_o} &= -2\Sigma (9) \times (14) = 0.000585 \end{aligned}$$

Thus,

$$\left.\begin{aligned} \Delta l &= -0.1500 \\ \Delta l' &= -.1307 \\ \Delta \beta &= -.0035 \\ \Delta \beta' &= -.0084 \end{aligned}\right\}$$

The new values of the parameters are then

$$\left.\begin{aligned} l &= -1.3160 \\ l' &= 3.1393 \\ \beta &= .4581 \\ \beta' &= -.2534 \end{aligned}\right\} \tag{26a}$$

Using these values of the parameters as first approximations, it is found that the value of M corresponding to these values of parameters is

$$M = 0.008072$$

which is larger than the value of M corresponding to the parameters found by Prony's method. The implication here is that the first approximations found above were good. In general, however, it has been found that these approximations are not so fine and that several applications of the above method are required before M begins to increase. It should be noted that the increase in M does not imply that the value of each parameter given by equations (26) is better than the corresponding value displayed in equations (26a). It merely implies that over the entire range the curve obtained from the parameters (26) fits the data better than does the curve corresponding to the parameters (26a).

Since the value of M obtained after an application of the method of steepest descent along a tangent was larger than the value corresponding to the values obtained by Prony's method, the methods will be applied using the parameters obtained by Prony's method as first approximations.

The Method of Steepest Descent to a Minimum

The method of steepest descent to a minimum is applied in tables I and II. The first 19 columns of the calculation are identical with the calculations needed for the method of steepest descent along a tangent which are displayed in table I. The remaining columns and the sums which are required are shown in table II.

It follows from equations (4b), (17), (18), and (19) that if circled numbers refer to columns, then

$$\left.\begin{aligned}
M_o &= \Sigma\ (14)^2 = 0.0028 \\
M_{l_o} &= 2\ \Sigma\ (14) \times (15) = 0.0105 \\
M_{l'_o} &= -2\ \Sigma\ (14) \times (19) = 0.00911 \\
M_{\beta_o} &= 2\ \Sigma\ (8) \times (14) = 0.000246 \\
M_{\beta'_o} &= -2\ \Sigma\ (9) \times (14) = 0.000585 \\
M_{l_o l_o} &= 2\ \Sigma\ (20) \times (21) = 0.4097 \\
M_{l'_o l'_o} &= 2\ \Sigma\ (19)^2 - 2\ \Sigma\ (14) \times (20) = 0.3928 \\
M_{\beta_o \beta_o} &= 2\ \Sigma\ (8)^2 = 1.291 \\
M_{\beta'_o \beta'_o} &= 2\ \Sigma\ (9)^2 = 2.487 \\
M_{l_o l'_o} &= -2\ \Sigma\ (21) \times (22) = 0.03107 \\
M_{l_o \beta_o} &= 2\ \Sigma\ (21) \times (23) = 0.5665 \\
M_{l_o \beta'_o} &= -2\ \Sigma\ (21) \times (24) = -0.3231 \\
M_{l'_o \beta_o} &= -2\ \Sigma\ (14) \times (24) - 2\ \Sigma\ (18) \times (23) = 0.3960 \\
M_{l'_o \beta'_o} &= 2\ \Sigma\ (18) \times (24) - 2\ \Sigma\ (14) \times (23) = 0.7898 \\
M_{\beta_o \beta'_o} &= -2\ \Sigma\ (8) \times (9) = 0.2445
\end{aligned}\right\} \quad (27)$$

Applying equations (8a), (12a), and (13a), it may be seen that

$$\left.\begin{aligned}
\Delta l &= -0.0217 \\
\Delta l' &= -.0188 \\
\Delta \beta &= -.0005 \\
\Delta \beta' &= -.0012
\end{aligned}\right\}$$

or, applying equations (26), that the new approximations to the parameters become

$$\left.\begin{aligned} l_o &= -1.1877 \\ l'_o &= 3.2512 \\ \beta_o &= .4611 \\ \beta'_o &= -.2462 \end{aligned}\right\}$$

Booth's Method

Booth's method is applied in tables I, II, III, and IV. Applying equations (15a) and (16) and the sums found in tables II, III, and IV, it may be found that

$$F = 0.1419$$

or that

$$\left.\begin{aligned} \Delta l &= -0.0217 \\ \Delta l' &= -.0188 \\ \Delta \beta &= -.0005 \\ \Delta \beta' &= -.0012 \end{aligned}\right\}$$

Finally, it follows that the new approximations are

$$\left.\begin{aligned} l_o &= -1.1877 \\ l'_o &= 3.2512 \\ \beta_o &= .4611 \\ \beta'_o &= -.2462 \end{aligned}\right\}$$

The Relaxation Method

The relaxation method requires first the calculation of the 19 columns displayed in table I. Further calculation may be necessary, but it is well to stop here and calculate M_o, M_{l_o}, $M_{l'_o}$, M_{β_o}, and $M_{\beta'_o}$. If either M_{β_o} or $M_{\beta'_o}$ is the largest of the four derivatives, no more columns will have to be computed. If, however, $M_{l'_o}$ is the largest derivative, one more column must be found; whereas if M_{l_o} is the largest two more must be found. Referring to equations (27), it may be seen that M_{l_o} is the largest of the four derivatives. Calculating columns 20 and 21 of table II and using equations (20) and (27), it may be seen that

$$\Delta l = -0.0255$$

or that the new approximation to l is

$$l_o = -1.1915$$

The Taylor's Series Method

With circled numbers referring to columns in table I, it may be seen that equations (21) become

$$\left.\begin{aligned}
&\Delta l \, \Sigma \, (15)^2 - \Delta l' \, \Sigma (15) \times (19) + \Delta\beta \, \Sigma \, (8) \times (15) - \Delta\beta' \, \Sigma \, (9) \times (15) = \\
&\qquad - \Sigma \, (14) \times (15) \\
&- \Delta l \, \Sigma \, (15) \times (19) + \Delta l' \, \Sigma \, (19)^2 - \Delta\beta \, \Sigma \, (8) \times (19) + \Delta\beta' \, \Sigma \, (9) \times (19) = \\
&\qquad \Sigma (14) \times (19) \\
&\Delta l \, \Sigma \, (8) \times (15) - \Delta l' \, \Sigma \, (8) \times (19) + \Delta\beta \, \Sigma \, (8)^2 - \Delta\beta' \, \Sigma \, (8) \times (9) = \\
&\qquad - \Sigma (8) \times (14) \\
&- \Delta l \, \Sigma \, (9) \times (15) + \Delta l' \, \Sigma \, (9) \times (19) - \Delta\beta \, \Sigma \, (8) \times (9) + \Delta\beta' \, \Sigma \, (9)^2 = \\
&\qquad \Sigma (9) \times (14)
\end{aligned}\right\}$$

Inserting the values of these sums found in table I and solving the resulting equations,

$$\left.\begin{aligned}
\Delta l &= -0.2089 \\
\Delta l' &= -.1680 \\
\Delta\beta &= .1353 \\
\Delta\beta' &= .0139
\end{aligned}\right\}$$

Also, the new approximations are

$$\left.\begin{aligned}
l_o &= -1.3749 \\
l'_o &= 3.1020 \\
\beta_o &= .5969 \\
\beta'_o &= -.2311
\end{aligned}\right\}$$

Final Values of the Parameters

Calculation of l, l', β, and β'.- Repeated application of any of the methods yields the following values of the parameters to about three significant figures:

$$\left.\begin{aligned} l &= -1.366 \\ l' &= 3.071 \\ \beta &= .6141 \\ \beta' &= -.2083 \end{aligned}\right\} \tag{28}$$

These values of the parameters give, finally,

$$M = 0.000895 \tag{29}$$

Calculation of the stability parameters.- The stability parameters b, k, C_1, and C_o of the test airplane from which the data of the previous example were measured will now be calculated. Using the values of l and l' given in equations (28) and applying equations (23), it may be seen that

$$\left.\begin{aligned} b &= 2.732 \\ k &= 11.30 \end{aligned}\right\}$$

In order to find C_1 and C_o by the method described, it is necessary to find the functions $e^{-lt}\delta(t)\cos l't$ and $e^{-lt}\delta(t)\sin l't$ and to integrate with respect to t to find σ_k and σ_k', respectively. The functions σ_k and σ_k' were found by means of a planimeter, utilizing figure 1 (see table V). Applying equations (25), and allowing the circled numbers to refer to the columns in table V, it is seen that

$$19 \times \frac{C_o}{l'} - 28\,\frac{C_1}{l'} = 33$$

Applying the least-squares principle, the following equations are obtained:

$$\left.\begin{aligned} \frac{C_o}{l'} \times \Sigma\ 19^2 - \frac{C_1}{l'}\,\Sigma\ 19 \times 28 &= \Sigma\ 19 \times 33 \\ -\frac{C_o}{l'}\,\Sigma\ 19 \times 28 + \frac{C_1}{l'}\,\Sigma\ 28^2 &= -\Sigma\ 28 \times 33 \end{aligned}\right\}$$

Thus, utilizing the sums displayed in table V,

$$\left.\begin{aligned} \frac{C_0}{l'} &= -12.34 \\ \frac{C_1}{l'} &= 5.17 \end{aligned}\right\}$$

or, recalling that $l' = 3.071$,

$$\left.\begin{aligned} C_0 &= -37.90 \\ C_1 &= 15.88 \end{aligned}\right\}$$

EVALUATION OF THE METHODS AND DEVICES FOR ACCELERATING CONVERGENCE

Before evaluating the methods, it is necessary to give the time required to apply each of them. The following table presents the number of hours required for one application of the corresponding method, including a check. The time required for a single application without the check is roughly three-fourths of the total time given in the following table:

Time Required for a Single Application of the Methods	
Method	Time required (hours)
Steepest descent along a tangent	3
Booth's method	8
Steepest descent to a minimum	$6\frac{1}{2}$
Taylor's series method	$5\frac{1}{2}$
Relaxation method	$3\frac{1}{4}$

Two applications of the Taylor's series method gave a value of $M = 0.000895$. Three applications of the method of steepest descent to a minimum and of Booth's method are required to obtain correspondingly small values of M. The value of M obtained after three applications of Booth's method is slightly smaller than that given by the other two methods; however, as is indicated by the following evaluation, it is not believed that this small added benefit merits the additional labor involved in Booth's method. The total time required for the Taylor's series method to bring the value of M down to the value given by equation (29) is 11 computer hours; for the method of steepest descent to a minimum, 19 hours were needed; whereas Booth's method required 24 hours.

The time of 3-1/4 hours given above for a single application of the relaxation method does not tell the complete story. It should be realized that in a second iteration by means of the relaxation method many of the columns and sums which are needed are identical with the corresponding columns or sums in the previous iteration and thus need not be computed. For example, in fitting a sum of two exponentials, if one iteration changes β' (say), only 7 out of 21 columns must be recalculated at the next iteration. Thus, although 10 iterations are required to reduce the value of M to approximately the value given in equation (29), 20 hours are required for the job.

Two further points should be made before an evaluation is applied to the above methods. First, it should be noted that the amount of work required to solve a given problem by the method of steepest descents increases very rapidly with m (where, as before, m is the number of parameters considered in the problem), due to the necessity for calculating all second derivatives of M with respect to the parameters. The rate of increase of hours of labor for the Taylor's series method, while not as rapid as for this last method, is still quite high. On the other hand, the amount of labor required to apply the relaxation method or Booth's method increases relatively slowly with m.

Finally, it should be realized that there are many devices available for accelerating the convergence of the iteration process. These devices become evident as familiarity with the methods increases. It is important to understand that any method, however irregular, is to be considered applicable provided only that the value of M is made to decrease. Such a device, for example, is the following. Suppose repeated application of one of the methods shows that the sign of the increments to be added to one of the parameters is always the same. A device which is often applicable in this case would be to increase the value of the increment found at some iteration. Conversely, if the signs of some increments change at successive iterations, it may be found advisable to use an increment not quite as large as that indicated by the method used. For a further discussion of these accelerating devices, see reference 8.

Due to the fact that the relaxation method varies only one parameter at a time, the computer is more aware of what is being accomplished at each iteration when applying this method than when applying the other methods. This, in turn, makes it easier to devise and apply "tricks" like the two described above for accelerating convergence when applying the relaxation method than when applying the other methods. This is particularly true when the number of parameters is large.

Taking all these factors into account the following conclusions would appear to be valid:

1. If the number of parameters is less than five or six, the Taylor's series method seems very well adapted to solving the problem. The method of steepest descent to a minimum also may be applied successfully, but more time is required for the minimization. Further, the type of computations used in the method of steepest descent to a minimum, being difficult to systematize, is peculiarly subject to numerical error on the part of the computer. Both the relaxation method and Booth's method also require more time than the Taylor's series method and are not as well suited to the solution of the problem as either of the other two methods.

2. If the number of parameters exceeds six, either Booth's method or the relaxation method may be applied successfully. Since Booth's method varies all of the parameters together rather than changing only one at a time as does the relaxation method, the former is to be preferred, except if it is desired to allow the computer to apply his own discretion to increase the rapidity of the convergence.

Only one more thing remains: the evaluation of the method of damped least squares. It should be remembered that this method was devised for the purpose of solving problems for which the Taylor's series method fails to converge. In the application of this method, the weighting factors w and w_k which occur in equation (4d) must be found at each iteration, thus adding further calculations to an already imposing array. Also, there are several devices which may be applied which remove the need for such a method with its attendant complications. First, one of the other methods described may not fail to converge and may be applied in place of the Taylor's series method. In particular, the method of steepest descent along a tangent may be used followed by applications of the Taylor's series method. Second, a new first approximation may be found from which the Taylor's series method or one of the other methods may converge. Finally, one of the devices mentioned above for increasing the rapidity of convergence may often do more than this: these methods frequently will cause a method to converge where it diverged before. It is believed that intelligent use of these three devices can take the place of the method of damped least squares.

CONVERGENCE TO A MERE STATIONARY POINT AS OPPOSED TO A TRUE MINIMUM

To review, what has been done thus far is the following: a point P, $(x_1^{(o)}, x_2^{(o)} \ldots, x_m^{(o)})$ has been found such that

$$\frac{\partial M}{\partial x_k}\left(x_1^{(o)}, x_2^{(o)} \ldots, x_m^{(o)}\right) = 0, \; k = 1, 2, \ldots, m$$

The possibility that higher derivatives of M are also zero at P has not entered the discussion. A point at which all the first and second derivatives of M are zero may be called a stationary point, as opposed to a true extremum (maximum or minimum) at which some of the second derivatives of M are not zero. (If M were a function of one variable, say x, alone, a stationary point would be a point of inflection at which $dM/dx = 0$.) This section will be devoted to a criterion for determining whether the iteration processes have converged to a true minimum or to a stationary point.

Let $M^{(o)}$ be the value of M at $(x_1^{(o)}, x_2^{(o)}, \ldots, x_m^{(o)})$. Let M_{ij} be the value of $\dfrac{\partial^2 M}{\partial x_i \partial x_j}$ at $(x_1^{(o)}, \ldots, x_m^{(o)})$. In reference 4, Booth shows that if $M^{(o)}$ is a true minimum, the quadratic form

$$Q = \sum_{i,j=1}^{m} M_{ij} x_i x_j$$

is positive definite, while it is well known (see, e.g., reference 10, p. 137) that the form Q is positive definite if and only if the following conditions hold:

$$\left.\begin{array}{r} M_{11} > 0 \\[4pt] \begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix} > 0 \\[4pt] \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix} > 0 \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \left\|M_{ij}\right\|_{i,j=1,\ldots,m} > 0 \end{array}\right\} \qquad (30)$$

In general, for curve-fitting problems conditions (30) do not have to be applied since the curve $q = q(t)$ may be plotted and compared with the curve $q = q_{\epsilon}(t)$, and it will usually be made clear whether or not a minimum has been reached. There are cases, however, where this is not so clear; in these cases, the conditions (30) may be important.

If it has been discovered that the minimization procedure has led to a stationary point rather than a true minimum, the means by which this difficulty may be overcome are not straightforward and a good deal of ingenuity may be required. The problem is, actually, similar to the problem of increasing M which was discussed above. Such devices as finding new first approximations, applying one of the other methods, etc., may be used. In particular, when finding new first approximations, choosing them farther in the same direction as the parameters were going when converging from the old approximations to the stationary point will often be fruitful.

USES OF THE METHODS OTHER THAN CURVE FITTING

The problem of curve fitting a nonlinear function has been discussed at length, but it has hardly been mentioned that there are other important uses for the methods discussed herein. These applications will be briefly discussed in the present section.

The first application is evident; the method may be applied to any minimization problem besides curve fitting. Such a problem arises in connection with guided missiles, for example. There are certain parameters of the missile and its control system (servo gain, time constants, etc.) which are susceptible to adjustment; such an adjustment is sought which minimizes the miss distance.

A second type of problem is one which frequently arises in applied mathematics. It may be generally described by saying that it includes any problem which may be reduced to a minimization problem. For example, consider the set of equations

$$\left.\begin{aligned} \varphi_1(x_1, \ldots, x_m) &= 0 \\ \varphi_2(x_1, \ldots, x_m) &= 0 \\ \cdots\cdots\cdots\cdot \\ \varphi_n(x_1, \ldots, x_m) &= 0 \end{aligned}\right\} \tag{31}$$

which are to be solved simultaneously. (It should be noticed that m does not have to equal n. The definition given below of a "solution" of equations (31) includes the possibility $m = n$ as a special case.) Equations (31) may or may not be linear. This problem may be reduced to one of minimization by defining a solution of equations (31) as any set of values of $x_1, \ldots, x_m$ which minimize

$$M = \sum_{k=1}^{n} \varphi_k{}^2$$

This definition evidently includes the case where equations (31) have a solution in the classical sense. A numerical example of this type of problem follows.

Consider the set of two equations

$$\left.\begin{aligned} x_1 &= \sin x_2 \\ x_1 + x_2 &= 1 \end{aligned}\right\}$$

These may be solved by minimizing

$$M = (x_1 - \sin x_2)^2 + (x_1+x_2-1)^2$$

Letting $M_1 = \frac{\partial M}{\partial x_1}$, $M_2 = \frac{\partial M}{\partial x_2}$, we have

$$\left.\begin{aligned} M_1 &= 2(2x_1+x_2-1- \sin x_2) \\ M_2 &= 2\left(x_1+x_2-1-x_1 \cos x_2+ \frac{\sin 2x_2}{2}\right) \end{aligned}\right\}$$

Rather than obtain first approximations by some cumbersome graphical method, $x_1 = x_2 = 0$ may be arbitrarily taken as an approximation and the method of steepest descent along a tangent (equations (14)) applied.

If zero subscripts denote the value of M and its derivatives at the first approximation $x_{1_0} = x_{2_0} = 0$,

$$\left.\begin{aligned} M_0 &= 1 \\ M_{1_0} &= -2 \\ M_{2_0} &= -2 \end{aligned}\right\}$$

Applying equations (14),

$$\left.\begin{aligned} \Delta x_1 &= \frac{2}{8} = 0.25 \\ \Delta x_2 &= \frac{2}{8} = .25 \end{aligned}\right\}$$

Changing notation so that zero subscripts refer to the new approximations

$$\left.\begin{aligned} x_{1_0} &= 0.25 \\ x_{2_0} &= .25 \end{aligned}\right\}$$

we obtain

$$\left.\begin{aligned} M_0 &= (0.0026)^2 + (0.5)^2 = 0.25 \\ M_{1_0} &= 2(-0.4974) = -0.9948 \\ M_{2_0} &= 2(-0.0025) = -0.005 \end{aligned}\right\}$$

Therefore, again using equations (14),

$$\left.\begin{aligned} \Delta x_1 &= 0.2513 \\ \Delta x_2 &= .0013 \end{aligned}\right\}$$

and, again changing notation,

$$\left.\begin{aligned} x_{1_0} &= 0.5013 \\ x_{2_0} &= .2513 \end{aligned}\right\}$$

This method may be continued until M increases, and it may be followed by applications of one of the other methods.

Another common problem which may be solved is the solution of polynomial equations (or, for that matter, other types of equations). This problem is, of course, the special case of the preceding problem which occurs when $m = n = 1$. Thus, the equation

$$\varphi(x) = 0$$

may be solved for x by minimizing

$$M = [\varphi(x)]^2$$

CONCLUDING REMARKS

Several methods for curve fitting a nonlinear function by least squares have been described. If the number of parameters which may be varied in any problem to obtain the best fit for a set of data be denoted by m, the evaluation leads to the following conclusions:

1. In general, the method of steepest descent along a tangent should be first applied to the first approximations. One of the finer methods should then be used to improve the parametric values obtained by this method.

2. The method of steepest descent to a minimum, while theoretically quite good, is rather long, the length of time required for a single iteration increasing rapidly with m, the number of parameters. For this reason, the method may be applied successfully if m is not greater than four or five; it is too cumbersome for higher order systems.

3. The Taylor's series method is somewhat better than the method of steepest descents from the point of view of speed and accuracy. The method is very well suited to problems in which m is less than five or six. It has the disadvantage of requiring the solution of a set of m simultaneous equations at each iteration, and thus may grow awkward for large m.

4. The relaxation method requires less time for a single iteration than any of the other methods. However, not as much is achieved per iteration. More precisely, while the other methods described improve the values of all the parameters at each iteration, the relaxation method changes only one parameter at a time. This disadvantage also has a compensation, however, in that a greater feel for what is being accomplished at each step is to be had. That is, the computer may use more of his own discretion and intelligence to improve the convergence. The relaxation method is well adapted to systems with large (i.e., greater than 6) values of m.

5. The amount of labor required for Booth's method, along with the relaxation method, increases relatively slowly with m. Thus, either Booth's method or the relaxation method may be applied if the number of parameters is large.

6. One further method, the so-called method of damped least squares, is discussed in the body of the report. This system was developed for use in cases where the Taylor's series method failed to converge. It might seem, however, that other devices, finding new first approximations, application of one of the other methods, under- or over-relaxation, and so forth, eliminate the need for any such method.

These conclusions may thus be finally summed up as in the following table:

Method	Evaluation
Steepest descent along a tangent	A rapid method for improving first approximations. Should be used on all problems unless it is definitely known that these approximations are very good
Taylor's series method	This appears to be the best method to apply if the number of parameters is less than five or six
Steepest descent to a minimum	Rather cumbersome, but may be applied successfully if the number of parameters does not exceed four
Relaxation method	Useful if the number of parameters exceeds six
Booth's method	This method appears best if the number of parameters is very much greater than eight. If m is between six and eight, Booth's method and the relaxation method appear to be equally useful
Damped least squares	Devices described above would appear to eliminate the need for any such method

Ames Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Moffett Field, Calif., Oct. 17, 1951.

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TABLE I.- THE METHOD OF STEEPEST DESCENT ALONG A TANGENT

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Row	t_i	$l_o t_i$	$e^{(3)}$	$l_o' t_i$	cos (5)	sin (5)	(4)×(6)	(4)×(7)	(8)× β_o	(9)× β_o'	(10) - (11)	$q_\epsilon(t)$	(12) - (13)	(2)×(12)	(9)× β_o	(8)× β_o'	(16)+ (17)	(2)×(18)
1	0.4	-0.4664	0.6273	1.308	0.2598	0.9657	0.1630	0.6058	0.0752	-0.1484	0.224	0.224	0	0.0894	0.2796	-0.0399	0.2397	0.0959
2	.5	-.5830	.5582	1.635	-.0642	.9979	-.0358	.5570	-.0165	-.1365	.120	.120	0	.0600	.2571	.0088	.2659	.1330
3	.6	-.6996	.4968	1.962	-.3812	.9245	-.1894	.4593	-.0874	-.1125	.025	.020	.005	.0151	.2120	.0464	.2584	.1550
4	.7	-.8162	.4421	2.289	-.6580	.7530	-.2909	.3329	-.1343	-.0616	-.053	-.057	.004	-.0369	.1537	.0713	.2250	.1575
5	.8	-.9328	.3935	2.616	-.8651	.5017	-.3404	.1974	-.1571	-.0484	-.109	-.112	.003	-.0870	.0911	.0834	.1745	.1396
6	.9	-1.0494	.3501	2.943	-.9803	.1973	-.3432	.0691	-.1584	-.0169	-.142	-.148	.006	-.1274	.0319	.0841	.1160	.1044
7	1.0	-1.1660	.3116	3.270	-.9918	-.1281	-.3090	-.0399	-.1426	.0098	-.152	-.160	.008	-.1524	-.0184	.0757	.0573	.0573
8	1.1	-1.2826	.2773	3.597	-.8981	-.4398	-.2490	-.1220	-.1149	.0299	-.145	-.150	.005	-.1593	-.0563	.0610	.0047	.0052
9	1.2	-1.3992	.2468	3.924	-.7092	-.7050	-.1750	-.1740	-.0808	.0426	-.123	-.127	.004	-.1481	-.0803	.0429	-.0374	-.0449
10	1.3	-1.5158	.2196	4.251	-.4453	-.8954	-.0978	-.1966	-.0451	.0482	-.093	-.097	.004	-.1213	-.0908	.0240	-.0668	-.0868
11	1.4	-1.6324	.1955	4.578	-.1340	-.9910	-.0262	-.1937	-.0121	.0475	-.060	-.062	.002	-.0834	-.0894	.0064	-.0830	-.1162
12	1.5	-1.7490	.1739	4.905	.1915	-.9815	.0333	-.1707	.0154	.0418	-.026	-.032	.006	-.0396	-.0788	-.0082	-.0870	-.1305
13	1.6	-1.8656	.1548	5.232	.4965	-.8680	.0769	-.1344	.0355	.0329	.003	-.005	.008	.0042	-.0620	-.0188	-.0808	-.1293
14	1.7	-1.9822	.1378	5.559	.7491	-.6625	.1032	-.0913	.0476	.0224	.025	.017	.008	.0428	-.0421	-.0253	-.0674	-.1146
15	1.8	-2.0988	.1226	5.886	.9221	-.3869	.1130	-.0474	.0522	.0116	.041	.030	.011	.0731	-.0219	-.0277	-.0496	-.0893
16	1.9	-2.2154	.1091	6.213	.9975	-.0701	.1088	-.0076	.0502	.0019	.048	.036	.012	.0918	-.0035	-.0267	-.0302	-.0574
17	2.0	-2.3320	.0971	6.540	.9672	.2539	.0939	.0247	.0433	-.0061	.049	.035	.014	.0988	.0114	-.0230	-.0116	-.0232
18	2.1	-2.4486	.0864	6.867	.8344	.5512	.0721	.0476	.0333	-.0117	.045	.032	.013	.0945	.0220	-.0177	.0043	.0090
19	2.2	-2.5652	.0769	7.194	.6130	.7901	.0471	.0608	.0217	-.0149	.037	.027	.010	.0805	.0281	-.0115	.0166	.0365
20	2.3	-2.6818	.0684	7.521	.3269	.9451	.0224	.0646	.0103	-.0158	.026	.020	.006	.0600	.0298	-.0055	.0243	.0559
21	2.4	-2.7984	.0609	7.848	.0059	1.0000	.0004	.0609	.0002	-.0149	.015	.015	0	.0362	.0281	-.0001	.0280	.0672
22	2.5	-2.9150	.0542	8.175	-.3155	.9489	-.0171	.0514	-.0079	-.0126	.005	.011	-.006	.0118	.0237	.0042	.0279	.0698
23	2.6	-3.0316	.0482	8.502	-.6036	.7973	-.0291	.0384	-.0134	-.0094	-.004	.008	-.012	-.0104	.0177	.0071	.0248	.0645
24	2.7	-3.1482	.0429	8.829	-.8277	.5612	-.0355	.0241	-.0164	-.0059	-.011	.005	-.016	-.0284	.0111	.0087	.0198	.0535
25	2.8	-3.2648	.0382	9.156	-.9641	.2656	-.0368	.0101	-.0170	-.0025	-.015	.003	-.018	-.0406	.0047	.0090	.0137	.0384
26	2.9	-3.3814	.0340	9.483	-.9983	-.0583	-.0339	-.0020	-.0156	.0005	-.016	.001	-.017	-.0467	-.0009	.0083	.0074	.0215
27	3.0	-3.4980	.0303	9.810	-.9267	-.3757	-.0281	-.0114	-.0130	.0028	-.016	0	-.016	-.0474	-.0053	.0069	.0016	.0048
28	3.1	-3.6146	.0269	10.137	-.7569	-.6536	-.0204	-.0176	-.0094	.0043	-.014	0	-.014	-.0425	-.0081	.0050	-.0031	-.0096
29	3.2	-3.7312	.0240	10.464	-.5069	-.8620	-.0122	-.0207	-.0056	.0051	-.011	0	-.011	-.0342	-.0096	.0030	-.0066	-.0211

$\Sigma\ (14)^2 = 0.0028$

$\Sigma\ (14) \times (15) = 0.00523$

$\Sigma\ (14) \times (19) = -0.00455$

$\Sigma\ (8) \times (14) = 0.000123$

$\Sigma\ (9) \times (14) = -0.000292$

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TABLE II.- THE METHOD OF STEEPEST DESCENT TO A MINIMUM

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
← The first nineteen columns are given in table I →																			(2)×(15)	(12)+(14)	(2)×(19)	(2)×(8)	(2)×(9)
1																			0.0358	0.224	0.0384	0.0652	0.2423
2																			.0300	.120	.0665	-.0179	.2785
3																			.0091	.030	.0930	-.1136	.2756
4																			-.0258	-.049	.1103	-.2036	.2330
5																			-.0696	-.106	.1117	-.2723	.1579
6																			-.1147	-.136	.0940	-.3089	.0622
7																			-.1524	-.144	.0573	-.3090	-.0399
8																			-.1752	-.140	.0057	-.2739	-.1342
9																			-.1777	-.119	-.0539	-.2100	-.2088
10																			-.1577	-.089	-.1128	-.1271	-.2556
11																			-.1168	-.058	-.1627	-.0367	-.2712
12																			-.0594	-.020	-.1958	.0500	-.2561
13																			.0067	.011	-.2069	.1230	-.2150
14																			.0728	.033	-.1948	.1754	-.1258
15																			.1316	.052	-.1607	.2034	-.0853
16																			.1744	.060	-.1091	.2067	-.0144
17																			.1976	.063	-.0464	.1878	.0494
18																			.1985	.058	.0189	.1514	.1000
19																			.1771	.047	.0803	.1036	.1338
20																			.1380	.032	.1286	.0515	.1486
21																			.0869	.015	.1613	.0010	.1462
22																			.0295	-.001	.1745	-.0428	.1285
23																			-.0270	-.016	.1677	-.0757	.0998
24																			-.0767	-.027	.1445	-.0959	.0651
25																			-.1137	-.033	.1075	-.1030	.0283
26																			-.1354	-.033	.0624	-.0983	-.0058
27																			-.1422	-.032	.0144	-.0843	-.0342
28																			-.1318	-.028	-.0298	-.0632	-.0546
29																			-.1094	-.022	-.0675	-.0390	-.0662

$\Sigma (14)^2 = 0.0028$

$\Sigma (14)\times(15) = 0.00523$

$\Sigma (14)\times(19) = -0.00455$

$\Sigma (8)\times(14) = 0.000123$

$\Sigma (9)\times(14) = -0.000292$

$\Sigma (20)\times(21) = 0.2049$

$\Sigma (19)^2 = 0.2146$

$\Sigma (14)\times(20) = 0.01819$

$\Sigma (8)^2 = 0.6456$

$\Sigma (9)^2 = 1.2437$

$\Sigma (21)\times(22) = -0.01553$

$\Sigma (21)\times(23) = 0.2832$

$\Sigma (21)\times(24) = 0.1615$

$\Sigma (14)\times(24) = -0.00300$

$\Sigma (18)\times(23) = -0.1950$

$\Sigma (18)\times(24) = 0.4078$

$\Sigma (14)\times(23) = 0.0129$

$\Sigma (8)\times(9) = -0.1222$

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TABLE III.- BOOTH'S METHOD; CALCULATION OF M_1

1	2	3	4	5	6	7	8	9	10	11	12	13	14
Row	t	lt	$e^{(3)}$	$l't$	cos (5)	sin (5)	(4)×(6)	(4)×(7)	(8)× β	(9)× β'	(10) - (11)	q_ϵ	(12) - (13)
1	0.4	-0.5264	0.5907	1.2557	0.3099	0.9508	0.1831	0.5616	0.0839	-0.1423	0.226	0.224	0.002
2	.5	-.6580	.5179	1.5697	.0011	1.0000	.0006	.5179	.0003	-.1312	.131	.120	.011
3	.6	-.7896	.4540	1.8836	-.3077	.9515	-.1397	.4320	-.0640	-.1095	.046	.020	.026
4	.7	-.9212	.3980	2.1975	-.5865	.8099	-.2334	.3223	-.1069	-.0817	-.025	-.057	.032
5	.8	-1.0528	.3490	2.5114	-.8079	.5893	-.2820	.2057	-.1292	-.0521	-.077	-.112	.035
6	.9	-1.1844	.3059	2.8254	-.9504	.3110	-.2907	.0951	-.1332	-.0241	-.109	-.148	.039
7	1.0	-1.3160	.2682	3.1393	-1.0000	.0023	-.2682	.0006	-.1229	-.0002	-.123	-.160	.037
8	1.1	-1.4476	.2351	3.4532	-.9519	-.3065	-.2238	-.0721	-.1025	.0183	-.121	-.150	.029
9	1.2	-1.5792	.2061	3.7672	-.8107	-.5855	-.1671	-.1207	-.0765	.0306	-.107	-.127	.020
10	1.3	-1.7108	.1807	4.0811	-.5902	-.8073	-.1066	-.1459	-.0488	.0370	-.086	-.097	.011
11	1.4	-1.8424	.1584	4.3950	-.3122	-.9500	-.0495	-.1505	-.0227	.0381	-.061	-.062	.001
12	1.5	-1.9740	.1389	4.7090	-.0033	-1.0000	-.0005	-.1389	-.0002	.0352	-.035	-.032	-.003
13	1.6	-2.1056	.1218	5.0229	.3055	-.9522	.0372	-.1160	.0170	.0294	-.012	-.005	-.007
14	1.7	-2.2372	.1068	5.3368	.5847	-.8112	.0624	-.0866	.0286	.0219	.007	.017	-.010
15	1.8	-2.3688	.0936	5.6507	.8066	-.5912	.0755	-.0533	.0346	.0140	.021	.030	-.009
16	1.9	-2.5004	.0821	5.9647	.9497	-.3132	.0780	-.0257	.0357	.0065	.029	.036	-.007
17	2.0	-2.6320	.0719	6.2786	1.0000	-.0045	.0719	-.0003	.0329	.0001	.033	.035	-.002
18	2.1	-2.7636	.0624	6.5925	.9526	.3044	.0594	.0190	.0272	-.0048	.032	.032	0
19	2.2	-2.8952	.0553	6.9065	.8120	.5837	.0449	.0323	.0206	-.0082	.029	.027	.002
20	2.3	-3.0268	.0485	7.2204	.5920	.8059	.0287	.0391	.0131	-.0099	.023	.020	.003
21	2.4	-3.1584	.0425	7.5343	.3143	.9493	.0134	.0403	.0061	-.0102	.016	.015	.001
22	2.5	-3.2900	.0373	7.8483	.0058	1.0000	.0002	.0373	.0001	-.0095	.010	.011	-.001
23	2.6	-3.4210	.0327	8.1622	-.3034	.9529	-.0099	.0312	-.0045	-.0079	.003	.008	-.005
24	2.7	-3.5532	.0286	8.4761	-.5827	.8127	-.0167	.0232	-.0077	-.0059	-.002	.005	-.007
25	2.8	-3.6848	.0251	8.7900	-.8052	.5930	-.0202	.0149	-.0093	-.0038	-.006	.003	-.009
26	2.9	-3.8164	.0220	9.1040	-.9490	.3153	-.0209	.0069	-.0096	-.0017	-.008	.001	-.009
27	3.0	-3.9480	.0193	9.4179	-1.0000	.0068	-.0193	.0001	-.0088	0	-.009	0	-.009
28	3.1	-4.0796	.0169	9.7318	-.9532	-.3022	-.0161	-.0051	-.0074	.0013	-.009	0	-.009
29	3.2	-4.2112	.0148	10.0458	-.8133	-.5818	-.0120	-.0086	-.0055	.0022	-.008	0	-.008

$\Sigma\ (14)^2 = 0.0081$

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TABLE IV.- BOOTH'S METHOD; CALCULATION OF $M_{1/2}$

1	2	3	4	5	6	7	8	9	10	11	12	13	14
Row	t	lt	$e^{(3)}$	$l't$	cos (5)	sin (5)	(4)×(6)	(4)×(7)	(8)× β	(9)× β'	(10) - (11)	q_ϵ	(12) - (13)
1	0.4	-0.4964	0.6087	1.2818	0.2850	0.9585	0.1735	0.5834	0.0798	-0.1454	0.225	0.224	0.001
2	.5	-.6205	.5377	1.6023	-.0316	.9995	-.0170	.5374	-.0078	-.1339	.126	.120	.006
3	.6	-.7446	.4749	1.9228	-.3448	.9387	-.1637	.4458	-.0753	-.1111	.036	.020	.016
4	.7	-.8687	.4195	2.2432	-.6229	.7823	-.2613	.3282	-.1201	-.0818	-.038	-.057	.019
5	.8	-.9928	.3705	2.5637	-.8376	.5462	-.3103	.2024	-.1427	-.0504	-.092	-.112	.020
6	.9	-1.1169	.3273	2.8841	-.9671	.2546	-.3165	.0833	-.1455	-.0208	-.125	-.148	.023
7	1.0	-1.2410	.2891	3.2046	-.9980	-.0630	-.2885	-.0182	-.1327	.0045	-.137	-.160	.023
8	1.1	-1.3651	.2554	3.5251	-.9274	-.3741	-.2369	-.0955	-.1089	.0238	-.133	-.150	.017
9	1.2	-1.4892	.2256	3.8455	-.7623	-.6472	-.1720	-.1460	-.0791	.0364	-.116	-.127	.011
10	1.3	-1.6133	.1992	4.1660	-.5197	-.8544	-.1035	-.1702	-.0476	.0424	-.090	-.097	.007
11	1.4	-1.7374	.1760	4.4864	-.2241	-.9746	-.0394	-.1715	-.0181	.0427	-.061	-.062	.001
12	1.5	-1.8615	.1554	4.8069	.0945	-.9955	.0147	-.1547	.0068	.0386	-.032	-.032	0
13	1.6	-1.9856	.1373	5.1274	.4032	-.9151	.0554	-.1256	.0255	.0313	-.006	-.005	-.001
14	1.7	-2.1097	.1213	5.4478	.6709	-.7415	.0814	-.0899	.0374	.0224	.015	.017	-.002
15	1.8	-2.2338	.1071	5.7683	.8704	-.4924	.0932	-.0527	.0429	.0131	.030	.030	0
16	1.9	-2.3578	.0946	6.0887	.9812	-.1932	.0928	-.0183	.0427	.0046	.038	.036	.002
17	2.0	-2.4820	.0836	6.4092	.9921	.1257	.0829	.0105	.0381	-.0026	.041	.035	.006
18	2.1	-2.6061	.0738	6.7297	.9020	.4318	.0666	.0319	.0306	-.0079	.039	.032	.007
19	2.2	-2.7302	.0652	7.0501	.7201	.6939	.0470	.0452	.0216	-.0113	.033	.027	.006
20	2.3	-2.8543	.0576	7.3706	.4648	.8854	.0268	.0510	.0123	-.0127	.025	.020	.005
21	2.4	-2.9784	.0509	7.6910	.1623	.9867	.0083	.0502	.0038	-.0125	.016	.015	.001
22	2.5	-3.1025	.0449	8.0115	-.1570	.9876	-.0070	.0443	-.0032	-.0110	.008	.011	-.003
23	2.6	-3.2266	.0397	8.3320	-.4600	.8879	-.0183	.0352	-.0084	-.0088	0	.008	-.008
24	2.7	-3.3507	.0350	8.6524	-.7163	.6978	-.0251	.0244	-.0115	-.0061	-.005	.005	-.010
25	2.8	-3.4748	.0310	8.9729	-.8996	.4366	-.0279	.0135	-.0128	-.0034	-.009	.003	-.012
26	2.9	-3.5989	.0274	9.2933	-.9914	.1311	-.0272	.0036	-.0125	-.0009	-.012	.001	-.013
27	3.0	-3.7230	.0242	9.6138	-.9822	-.1879	-.0238	-.0045	-.0109	.0011	-.012	0	-.012
28	3.1	-3.8471	.0213	9.9343	-.8730	-.4877	-.0186	-.0104	-.0086	.0026	-.011	0	-.011
29	3.2	-3.9712	.0189	10.2547	-.6750	-.7379	-.0128	-.0139	-.0059	.0035	-.009	0	-.009

$\Sigma \; (14)^2 = 0.0036$

TABLE V.- COMPUTATIONS REQUIRED FOR THE CALCULATION OF C_1 AND C_0

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Row	t_k	$\delta(t_k)$	$e^{-l t_k}$	$\cos l' t_k$	$\sin l' t_k$	4 × 5	4 × 6	3 × 7	3 × 8	σ_k	σ_k'	$\sigma - \sigma_k$	$\sigma' - \sigma_k'$	$\frac{5}{4}$	$\frac{6}{4}$	14 × 15	13 × 16
1	0.	0	1.000	1.000	0	1.000	0	0	0	0.	0	0.02172	0.01498	1.000	0	0.01498	0.
2	.05	.005	1.071	.988	.153	1.058	.164	.005	.001	.00005	.00001	.02167	.01497	.923	.143	.01382	.00310
3	.10	.045	1.146	.953	.302	1.093	.347	.049	.016	.00128	.00030	.02044	.01468	.832	.264	.01221	.00540
4	.15	.105	1.227	.896	.445	1.099	.546	.115	.057	.00534	.00220	.01638	.01278	.730	.363	.00933	.00595
5	.20	.125	1.314	.817	.576	1.074	.757	.134	.095	.01176	.00598	.00996	.00900	.622	.438	.00560	.00436
6	.25	.100	1.407	.719	.695	1.012	.977	.101	.098	.01812	.01088	.00360	.00410	.511	.494	.00210	.00178
7	.30	.030	1.507	.605	.796	.911	1.200	.027	.036	.02132	.01422	.00040	.00076	.401	.528	.00030	.00021
8	.35	0	1.613	.476	.880	.768	1.419	0	0	.02172	.01498	0	0	.295	.546	0	0
9	.40	0	1.727	.336	.942	.580	1.627	0	0	.02172	.01498	0	0	.195	.545	0	0

	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
Row	17 - 18	13 × l'	14 × l	20 - 21	15 × 22	14 × l'	13 × l	24 + 25	16 × 26	23 + 27	15 × β	16 × β'	29 - 30	$q_e(t_k)$	32 - 31
1	0.01498	0.06670	-0.02046	0.08716	0.08716	0.04600	-0.02967	0.01633	0	0.08716	0.614	0	0.614	0.	-0.614
2	.01072	.06655	-.02045	.08700	.08030	.04597	-.02960	.01637	.00234	.08264	.567	-.030	.597	.008	-.589
3	.00681	.06277	-.02005	.08282	.06891	.04508	-.02792	.01716	.00453	.07344	.511	-.055	.566	.095	-.471
4	.00338	.05030	-.01746	.06776	.04946	.03925	-.02238	.01687	.00612	.05558	.448	-.076	.524	.200	-.324
5	.00124	.03059	-.01229	.04288	.02667	.02764	-.01361	.01403	.00615	.03282	.382	-.091	.473	.310	-.163
6	.00032	.01106	-.00560	.01666	.00851	.01259	-.00492	.00767	.00379	.01230	.314	-.103	.417	.360	-.057
7	.00009	.00123	-.00104	.00227	.00091	.00233	-.00055	.00178	.00094	.00185	.246	-.110	.356	.320	-.036
8	0	0	0	0	0	0	0	0	0	0	.181	-.114	.295	.272	-.023
9	0	0	0	0	0	0	0	0	0	0	.120	-.114	.234	.224	-.010

$\Sigma\ (19)^2 = 0.0003988$

$\Sigma\ (19)\ (28) = 0.00292$

$\Sigma\ (19)\ (33) = -0.02004$

$\Sigma\ (28)^2 = 0.02414$

$\Sigma\ (28)\ (33) = -0.1609$

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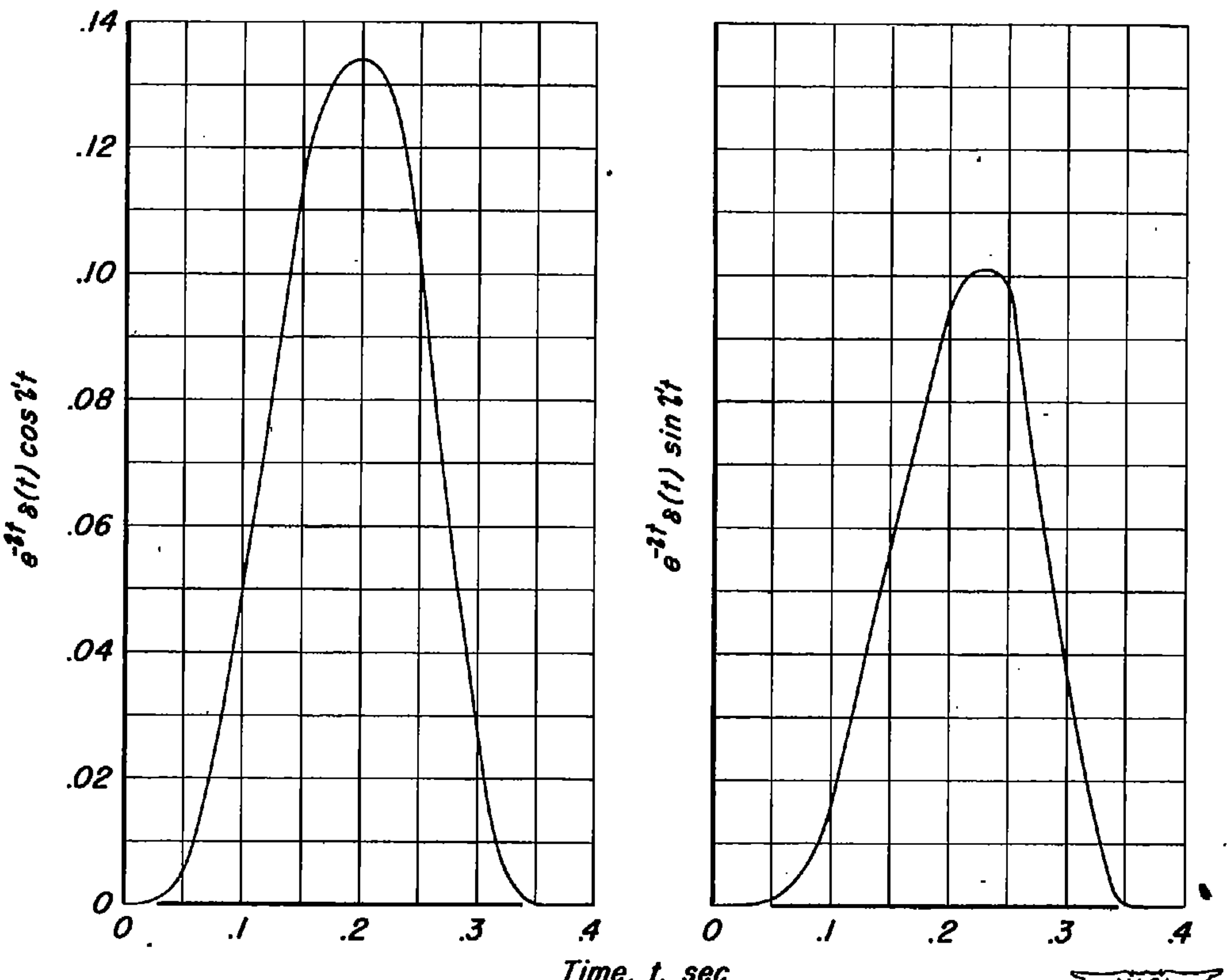

(a) $e^{-\ell t}\delta(t)\cos \ell' t$ (b) $e^{-\ell t}\delta(t)\sin \ell' t$

Figure 1.- The variation with time of two quantities required for the calculation of C_1 and C_0.

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